IMMERSED SURFACES AND DEHN SURGERY

Ying-Qing Wu[1]

§1. Introduction

The problem of how many Dehn filling on a torus boundary component T of a 3-manifold M will make a closed embedded essential surface F compressible has been settled. A slope β on T is a coannular slope if it is homotopic to some curve on F. As an embedded essential surface, F can have at most one coannular slope. If F has a coannular slope β on T, then by a result of Culler-Gordon-Luecke-Shalen [CGLS, Theorem 2.4.3], F is incompressible in all $M(\gamma)$ such that $\Delta(\beta, \gamma) > 1$, where $\Delta(\beta, \gamma)$ denotes the minimal intersection number between the slopes β and γ. If F has no coannular slopes, then it is incompressible in the Dehn filling space $M(\gamma)$ for all but at most three γ [Wu]. There are examples showing that these are the best possible.

While many manifolds do not contain embedded essential surfaces, it has been shown by Cooper, Long and Reid [CLR] that most bounded 3-manifolds, in particular all hyperbolic manifolds with some toroidal boundaries, contain immersed closed essential surfaces. There has been a lot of interest recently on immersed surfaces, see for example [AR,CLR,CL1,CL2,Oe,Re]. It seems important to understand to which extent the above theorems for embedded surfaces can be generalized to immersed surfaces.

Let S be a surface of finite type, i.e. compact surface with finitely many points removed. S may be disconnected or unorientable. We define a *surface* (of type S) in M to be a continuous piecewise smooth map $F : S \to M$ which is an immersion almost everywhere. F is *hyperbolic* if all components of F have negative Euler characteristic. A compact 3-manifold M is hyperbolic if its interior admits a complete hyperbolic structure.

Let T be a set of tori in ∂M. A curve on a surface is *simple* if it has no self-intersection. A *slope* γ_i on T is the isotopy class of a simple nontrivial curve on T. A

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slope γ is a*coannular slope* if some nontrivial multiple of γ is homotopic to a curve on F, in which case we say that F is coannular to T. A *multiple slope* $\gamma = (\gamma_1, ..., \gamma_n)$ on T is a set of slopes γ_i, one for each component T_i of T. Denote by $M(\gamma)$ the *Dehn filling space* along γ, i.e. the manifold obtained by attaching a solid torus V_i to each T_i $(i \leq n)$ so that γ_i bounds a meridian disk in V_i. Given two slopes α, β on a torus, we use $\Delta(\alpha, \beta)$ to denote the minimal geometric intersection number between α and β. If γ is a multiple slope and β is a finite set of slopes on T, denote

$$\Delta(\gamma, \beta) = \min \{ \Delta(\gamma_i, \beta_j) \mid \beta_j \subset T_i \}$$

In particular, $\Delta(\gamma, \beta) > 0$ if and only if $\gamma_i \notin \beta$ for all i. Note that β may have none or finitely many slopes on a component T_i of T. The following is our main theorem.

Theorem 5.3. *Let T be a set of tori on the boundary of a compact, orientable, hyperbolic 3-manifold W. Let F be a compact essential surface in W with $\partial F \subset \partial M - T$, and let β be the set of coannular slopes of F on T. Then there is an integer K and a finite set of slopes Λ on T, such that F is π_1-injective in $W(\gamma)$ for all multiple slopes γ on T satisfying $\Delta(\gamma, \beta) \geq K$ and $\gamma_i \notin \Lambda$.*

The result is best possible in the sense that there is no universal bound on the constant K, see Theorem 6.1. Note that F has only finitely many coannular slopes on T, i.e. β is a finite set. See the remark before Theorem 5.2. Thus in certain sense, Theorem 5.2 says that F survives most Dehn fillings on M. In particular, if F is not coannular to T, then F survives all surgeries after excluding a finite set of slopes on each component of T.

When T has only one component, Theorem 5.3 can be generalized to arbitrary compact orientable 3-manifolds M. However the theorem is no longer true when M contains some Seifert fibred submanifolds and T contains more than one components. An easy example is when T is a pair of tori T_1, T_2 coannular to each other. If F is compressible in $M(\gamma_1, \gamma_2)$, then it is compressible in $M(\gamma_1', \gamma_2')$ for all (γ_1', γ_2') obtained by twisting (γ_1, γ_2) along an essential annulus with one boundary component on each of T_i. More complicated examples can be constructed where no two components of T are coannular. However, the theorem is true if one further excludes all slopes of distance at most one from the fiber slopes. More details will appear elsewhere.

Another interesting topic is to construct immersed essential surfaces in hyperbolic 3-manifolds. See for example [AR, CLR, FF, CL1, CL2, Li]. One of the most important method is the Freedman tubing [FF]. Given a proper surface F in M, a *Freedman*

tubing $\hat{F}$ of F is a surface obtained from F by adding some annuli on ∂M with boundary on ∂F. This idea has been used in several important works, see [CLR, CL1, CL2, Li]. In particular, it was first proved by Cooper and Long [CL2] that a Freedman tubing of an embedded, geometrically finite surface is essential if the tubes are long enough. A combinatorial proof has been given by Li [Li], which also yields an upper bound of tube length in terms of genus and number of boundary components of F. They have also shown that the tubed surface survives most Dehn fillings, which, combined with a result of Culler and Shalen [CS], implies that all but finitely many Dehn filling spaces of a hyperbolic manifold contain an immersed surface.

Define the *wrapping number* wrap(A) of an annulus A on a torus T to be the minimum algebraic intersection number between A and all points of T. If $\hat{F}$ is a Freedman tubing of F, define $w(\hat{F}, F)$ to be the minimum of wrap(A_i) over all components A_i of $\hat{F} - F$. The following theorem generalizes the above result to immersed essential surfaces.

Theorem 5.7. *Let F be a geometrically finite surface in a compact hyperbolic 3-manifold W. Then there is a constant K such that if $\hat{F}$ is a Freedman tubing of F with wrap$(\hat{F}, F) \geq K$, then $\hat{F}$ is π_1-injective in W.*

The assumption that F be geometrically finite is necessary, otherwise F would be a virtual fiber, and hence no Freedman tubing of it would be essential. Immersed surfaces are much more abundant than embedded ones. For example, Oertel [Oe] and Maher [Ma] showed that in certain manifolds all slopes are realized as boundary slopes of immersed essential surfaces, while Hatcher [Ha] showed that there are only finitely many boundary slopes of embedded surfaces in these manifolds. More immersed surfaces can be obtained by projecting to M embedded surfaces in covering spaces of M. The boundary of such a surface may have several different slopes on the same torus component of ∂M. Our theorem applies to such surfaces as well, and there is no restriction on the orientability of F or $\hat{F}$. When ∂M is a set of tori and F is a proper surface, a essential Freedman tubing is automatically geometrically finite because it has accidental parabolics, hence by Theorem 1.1 it survivesill survive most Dehn fillings.

The idea of our proof is to use area estimation to show that certain curves in a negatively curved space are nontrivial. In section 2 we will use some results in minimal surface theory to show that if a piecewise geodesic curve α is trivial in a negatively curves 3-manifold M, then it bounds a disk whose intersection with the hyperbolic part of M has area bounded above by the total external angle of α. In

section 3 we give some estimation for areas of surfaces in truncated hyperbolic cusps, using integral of certain differential forms and Stokes theorem. These result will then be used in section 4 to show that curves in M satisfying certain conditions do not bound any disk, hence is nontrivial in M. The essentiality of surfaces in Dehn filling space and the essentiality of tubing surfaces in hyperbolic manifolds follow from these results by showing that all nontrivial curves on the surface satisfy those conditions. In section 6 we will show that there is no universal upper bounds for the bad fillings, and post several problems arisen in this research.

Definitions and conventions. All 3-manifolds in this paper are assumed orientable. Let $F : S \to M$ be a surface. A point $p \in S$ is a *regular point* if F is a local immersion at p. Otherwise it is a *singular point.* Almost all points of S are regular points since F is assumed to be an immersion almost everywhere. We will use F the same way as we would for embedded surfaces. Thus for example ∂F denotes the restriction of F to ∂S, and if N is a submanifold of M then $F \cap N$ denotes the restriction of F on $F^{-1}(N) \subset S$, which is considered as a subsurface of F if $F^{-1}(N)$ is a subsurface of S. By a *curve* on a surface $F : S \to M$ we mean the composition $F \circ \alpha$, where $\alpha : S^1 \to S$ is a closed curve on S. Similarly if $\alpha : I \to S$ is an arc then $F \circ \alpha$ is called an *arc* on F. We say that the arc $F \circ \alpha$ has endpoints on ∂F if $\partial\alpha \subset \partial S$, in which case it is a called a proper arc.

Given two arcs or curves α, β on F or in a manifold M, we use $\alpha \sim \beta$ to denote that α, β are homotopic. Homotopy of arcs and curves are different. Two arcs α, β are *homotopic* if they are homotopic rel boundary in the usual sense, while two curves are homotopic if they are freely homotopic. A curve in a space is *trivial* if it is null-homotopic. An arc α on a surface F is *essential* if it is not homotopic to an arc on ∂F.

A surface $F : S \to M$ is *incompressible* if any nontrivial curve on F is also nontrivial in M. Note that F is incompressible if and only if it is π_1*-injective*, that is, $F_* : \pi_1 S_i \to \pi_1 M$ is an injective map for all components S_i of S. A compact surface F is *proper* if $\partial F \subset \partial M$. F is ∂*-incompressible* if no essential arc of F is homotopic in M to an arc on ∂M. A proper surface F in M is *essential* if it is incompressible, ∂-incompressible, and is not rel ∂F homotopic to a surface on ∂M.

We refer the readers to [Th1] and [Mg] for basic concepts about hyperbolic 3-manifolds. In different sections below, M may denote either a complete hyperbolic manifold or a compact manifold with interior a complete hyperbolic manifold. If M is a complete hyperbolic manifold, the *injective radius* of a point x in M is the supremum

of radii of all embedded balls in M centered at x. Denote by $M_{(0,\epsilon]}$ the set of points which has injective radius at most ϵ, and by $M_{[\epsilon,\infty)}$ the set with injective radius at least ϵ. It is well known (see [Mg]) that when ϵ is sufficiently small, $M_{(0,\epsilon]}$ is a set of cusps, in which case we use $N = N_\epsilon$ to denote the toroidal cusp components of $M_{(0,\epsilon]}$, and $T = T_\epsilon$ the boundary tori of N.

The hyperbolic structure of M induces a Euclidean metric on $T = T_\epsilon$. If α is either a curve on T or an arc in M which is homotopic to an arc on T, then α can be homotoped to a geodesic α' on T. Define $t(\alpha)$ to be the Euclidean length of α', and call it the *T-length* of α. Notice that it depends only on ϵ and the homotopy class of α. If γ is another curve or arc on T, and γ' the geodesic on T homotopic to γ, then the *T-length of α relative to* γ, denoted by $t_\gamma(\alpha)$, is defined as

$$t_\gamma(\alpha) = t(\alpha)|\sin\theta|, \tag{1-1}$$

where θ is the angle between α' and γ'. Geometrically, $t_\gamma(\alpha)$ is the length of the orthogonal projection of α' to a line orthogonal to γ'. These notations will be used throughout the paper.

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§2. Minimal surfaces and the Plateau problem

Let $F : S \to M$ be a surface of type S in a Riemannian manifold M. Recall that F is assumed piecewise smooth. In this section we will also assume that F is oriented. If ω is a differential 2-form of M, then by the restriction of ω to F we mean the 2-form $F^*(\omega)$ on S defined on all smooth points of F, and the integral of ω on F is defined as

$$\int_F \omega = \int_S F^*(\omega).$$

Since F is piecewise smooth, this is well defined.

The Riemannian metric on M induces a Riemannian metric on the set of regular points of F, which determines a volume form ω_F. More explicitly, if (u_1, u_2) is a local coordinate system of S at a regular point p of F which is compatible with the orientation of S, then the tangent vectors $\partial_i = \frac{\partial}{\partial u_i} \in T_pS$ are mapped to $F_*(\partial_i)$ in $T_{F(p)}M$. The Riemannian metric of M determines an inner product $\langle \cdot, \cdot \rangle$ on $T_{F(p)}M$.

Let $g_{ij} = \langle F_*(\partial_i), F_*(\partial_j)\rangle$. Then

$$\omega_F = \sqrt{\det(g_{ij})}\ du_1 \wedge du_2.$$

This is a well-defined 2-form on S. Given a function $f(p)$ on S, which we consider as a function on F, the integral of f on F is defined as

$$\int_F f = \int_S f(p)\,\omega_F.$$

In particular, when $f = 1$, this defines the area of F:

$$\text{Area}(F) = \int_F 1 = \int_S \omega_F.$$

If M is of dimension two, then it has a volume form ω_M, in which case $\omega_F = \pm F^*(\omega_M)$, where the sign depends on whether F is orientation preserving or orientation reversing at that point. Given a 2-form ω on S with local presentation $\omega = \varphi\, du \wedge dv$, where (u, v) is a local coordinate system compatible with the orientation of S, we use $|\omega|$ to denote the 2-form $|\varphi| du \wedge dv$. Thus $\omega_F = |F^*(\omega_M)|$ when M is a surface.

We refer the readers to [Dc] for the definitions of curvatures and second fundamental form of submanifolds. Let (h_{ij}) be the second fundamental form of F at a regular point p, with respect to a basis (v_1, v_2) of $T_pF \subset T_pM$, then the Gauss formula (cf. [Dc, p.130]) says

$$K = \overline{K}(v_1, v_2) + \det(h_{ij}) = \overline{K}(v_1, v_2) + h_{11}h_{22} - h_{12}^2$$

where K is the curvature of F, and $\overline{K}$ is the sectional curvature of M. A continuous map $F : S \to M$ is a *minimal surface* if it is smooth in the interior of S, and its mean curvature $h_{11} + h_{22}$ is always zero. F is not required to be smooth on ∂S. Thus if F is a minimal surface then $h_{11}h_{22} \le 0$, so from the above we have $K \le \overline{K}(v_1, v_2)$. In particular, if M is a hyperbolic manifold, which by definition has constant sectional curvature $\overline{K} = -1$, then $K \le -1$.

The classical Plateau problem asks if a Jordan curve in $\mathbb{R}^n$ bounds a surface of disk type with minimal area. A solution to the Plateau problem is necessarily a minimal surface, which is harmonic in the interior of D, and is continuous on D. The problem was first solved by Douglas [Dg], and has been generalized by Morrey [Mr] to many Riemannian manifolds. The regularity of solutions has also been deeply studied. For our purpose, the following result suffices.

Lemma 2.1. *Let C be a null-homotopic, smooth, embedded circle in a complete, negatively curved 3-manifold M with hyperbolic ends. Then*

(i) C bounds a minimal surface $F : D^2 \to M$ of disk type, which minimizes the area of all disk type surfaces bounded by C;

(ii) F is a smooth map on D^2;

(iii) if K is the curvature function of F, and κ the geodesic curvature function of C in M, then

$$\int_F K + \int_C \kappa \geq 2\pi.$$

Proof. (i) This follows from Morrey's solution of the Plateau problem for Riemannian manifolds [Mr]. Morrey's result says that if M is a complete Riemannian manifold which is almost homogeneous, then any null-homotopic curve C in M bounds a minimal surface which minimizes the area of all disk type surfaces bounded by C. Since we have assumed that M is complete and has hyperbolic ends, Morrey's result applies.

(ii) This follows from Theorem 4 in Chapter 7 of [DHKW], which says that the degree of smoothness of a minimal surface on its boundary C is at least that of C and M. Since both C and M are assumed smooth, the result follows.

(iii) We need the following Gauss-Bonnet theorem for minimal surfaces with smooth boundary:

$$\int_F K + \int_{\partial F} \kappa_g = 2\pi + 2\pi \sum_{w \in \sigma'} \nu(w) + \pi \sum_{w \in \sigma''} \nu(w),$$

where κ_g is the geodesic curvature of ∂F on F, σ' and σ'' the set of interior and boundary branch points, respectively, and $\nu(w)$ the branch index of w, which is non-negative. For minimal surfaces in $\mathbb{R}^n$, this is Theorem 1 in Chapter 7 of [DHKW], and for minimal surfaces in Riemannian manifolds it is proved by Kaul [K]. The proof would be easy if one knows the smoothness of F on its boundary, which is (ii) above, and the local behavior of F near its branch points, which was done by Heinz and Hildebrandt [HH]. If p is a regular point of F on ∂F, $\mathbf{n}$ the principal normal vector of ∂F at p, $\mathbf{n}'$ the inward normal vector of p on F, and θ the angle between $\mathbf{n}$ and $\mathbf{n}'$, then $\kappa_g = (\cos\theta)\,\kappa$. Therefore we have $\kappa_g \leq \kappa$, and the result follows. $\square$

If C is a piecewise geodesic curve, and p is a corner point of C, then by going around C in a certain direction, we get two tangent vectors v_1, v_2 at p. The *external angle* of C at p is the angle between v_1 and v_2. The *total external angle* of C is the sum of external angles at all the corner points of C.

Proposition 2.2. *Let M be a complete negatively curved 3-manifold with hyperbolic ends, and let M_h be a hyperbolic submanifold of M. Suppose C is a piecewise geodesic in M such that M is hyperbolic near all corners of C. Let Θ be the total external angle of C. Then C bounds a surface F of disk type in M such that*

$$\text{Area}(F \cap M_h) \leq \Theta - 2\pi.$$

Proof. At each corner p, let $D_p = \exp D_\delta$, where D_δ is a disk of radius δ on the plane in T_pM containing the two tangent vectors of C at p, and exp the exponential map. Since M is hyperbolic near p, by choosing δ small enough we may assume that D_p is an embedded totally geodesic disk in M. Let α_1', α_2' be the two geodesic segment of $C \cap D_p$. Choose a point p_i in the interior of each α_i', and let α_i be the subarc of α_i' connecting p_i to p. Connect p_1 to p_2 by a smooth arc γ_p such that $(C - \alpha_1 \cup \alpha_2) \cup \gamma_p$ is smooth in D_p, and γ_p is concave on the region Δ_p bounded by $\alpha = \alpha_1 \cup \alpha_2 \cup \gamma_p$. See Figure 2.1.

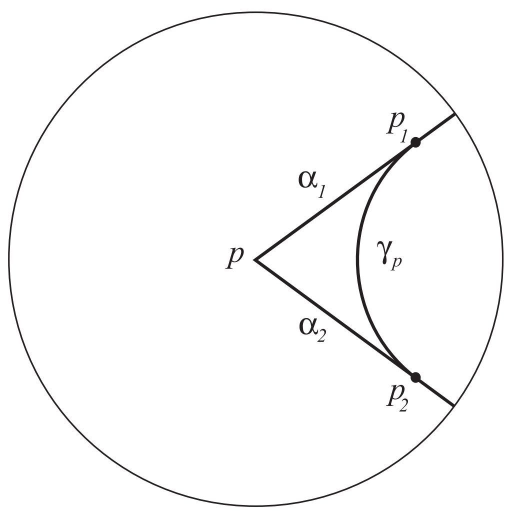

Figure 2.1

Since D_p is totally geodesic, the curvature κ of γ_p in M is the same as that in D_p. Since it is concave as a boundary curve of Δ_p, its curvature κ_g as boundary curve of Δ_p is $-\kappa$. The total external angle of $\partial \Delta_p$ is $2\pi + \theta(p)$, where $\theta(p)$ is the external angle of C at p. Therefore by the Gauss-Bonnet theorem applied to Δ_p, we have

$$\int_{\Delta_p} (-1) + \int_{\partial \Delta_p} \kappa_g + (2\pi + \theta(p)) = 2\pi.$$

The first integral is $-\text{Area}(\Delta_p)$, and the second equals $-\int_{\gamma_p} \kappa$. Hence

$$\text{Area}(\Delta_p) + \int_{\gamma_p} \kappa = \theta(p).$$

Let C' be the smooth curve obtained from C by replacing $\alpha_1 \cup \alpha_2$ with γ_p at each corner p, and let F' be the minimal surface bounded by C' as given in Lemma 2.1. Then $F = F' \cup (\cup \Delta_p)$ is a surface bounded by C. Since the curvature K of F' satisfies $K \leq -1$ in M_h and $K < 0$ elsewhere, by Lemma 2.1(3) and the above we have

$$\begin{aligned}
\text{Area}(F \cup M_h) &\leq \sum_p \text{Area}(\Delta_p) + \text{Area}(F' \cap M_h) \leq \sum_p \text{Area}(\Delta_p) - \int_{F'} K \\
&\leq \sum_p \text{Area}(\Delta_p) + \int_{\partial F'} \kappa - 2\pi = \sum_p \left[\text{Area}(\Delta_p) + \int_{\gamma_p} \kappa \right] - 2\pi \\
&= \sum \theta(p) - 2\pi = \Theta - 2\pi.
\end{aligned}$$

□

Remark 2.3. (1) Charles Frohman pointed out that when M is hyperbolic, Proposition 2.2 can be proved easily by considering a disk bounded by C which is a union of totally geodesic triangles. Thus the above proof using minimal surface theory is necessary only if M is negatively curved but not hyperbolic.

(2) Proposition 2.2 would follow more directly if we had a Gauss-Bonnet type formula for minimal surfaces with boundary a smooth curve with corners. It should look like:

$$\int_F K + \int_{\partial F} \kappa_g + \sum_{p \in \sigma'''} (\pm \theta_p) = 2\pi + 2\pi \sum_{w \in \sigma'} \nu(w) + \pi \sum_{w \in \sigma''} \nu(w)$$

where σ''' is the set of corner points, and θ_p the external angle of C at p. Note that negative sign could appear before θ_p if p is a branch point. The formula could be proved in the usual way if we know the local behavior of F near the corner points, which was done in Chapter 8 of [DHKW] in the special case that F is in Euclidean space. Unfortunately I cannot find a reference for either the formula or the local behavior near corners of a minimal surface F in a Riemannian manifold.

3. Area estimation for surfaces in truncated hyperbolic cusps

Throughout this paper, we will always consider the hyperbolic space $\mathbb{H}^3$ as in the upper half space model. Denote by $\mathbb{H}^3_1$ the hyperbolic horoball $\{(x, y, z) \mid z \geq 1\}$. For

$b > 1$, denote by $\mathbb{H}^3_{1,b}$ the subset of $\mathbb{H}^3_1$ where $z \leq b$. Consider $\mathbb{H}^2$ as the subset of $\mathbb{H}^3$ corresponding to the yz-plane. Define $\mathbb{H}^2_1 = \mathbb{H}^3_1 \cap \mathbb{H}^2$, and $\mathbb{H}^2_{1,b} = \mathbb{H}^3_{1,b} \cap \mathbb{H}^2$. For simplicity, we use (y, z) to denote a point $(0, y, z)$ in $\mathbb{H}^2$.

Consider the following subset $R_1(a, b)$ and $R_2(a, b)$ of $\mathbb{H}^2_1$ as shown in Figure 2.1, where $R_1(a, b)$ is a Euclidean rectangle, and $R_2(a, b)$ is the intersection with $\mathbb{H}^2_{1,b}$ of a Euclidean disk which is centered at the origin and intersects the horizontal line at $z = 1$ in an arc of length a. Thus it has radius $\sqrt{1 + (a/2)^2}$. More explicitly, we have

$$R_1(a, b) = \{(y, z) \in \mathbb{H}^2 \mid 0 \leq y \leq a,\ 1 \leq z \leq b\},$$
$$R_2(a, b) = \{(y, z) \in \mathbb{H}^2 \mid 1 \leq z \leq b,\ y^2 + z^2 \leq 1 + (\frac{a}{2})^2\}.$$

Define a function $\eta(x)$ by

$$\eta(x) = x - 2 \arctan \frac{x}{2}.$$

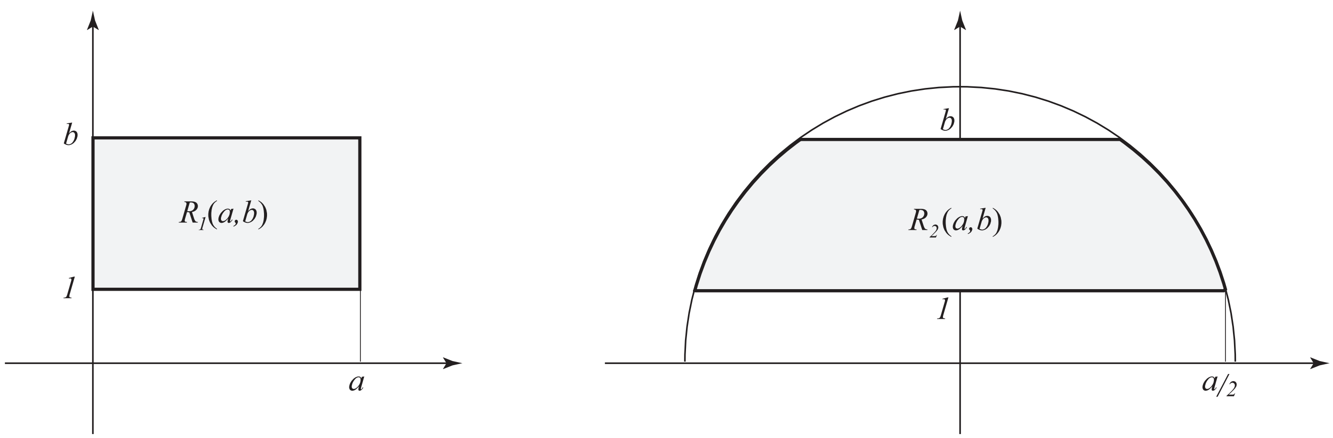

Figure 3.1

Lemma 3.1. *(1)* Area$(R_1(a, b)) = a(1 - 1/b)$.

(2) Area$(R_2(a, b)) = \begin{cases} \eta(a) & b^2 \geq 1 + \frac{a^2}{4} \\ \eta(a) - \eta(\frac{\sqrt{1+a^2/4-b^2}}{b}) & b^2 \leq 1 + \frac{a^2}{4} \end{cases}$

(3) If $a \geq 3\pi$ and $b \geq 5$, then Area$(R_2(a, b)) > 2\pi$.

Proof. These would follow from the Gauss-Bonnet theorem and the fact that a horizontal line in $\mathbb{H}^2$ at height b has curvature $1/b$, with normal vector pointing upward. The following is a direct calculation.

(1) Area$(R_1(a, b)) = \int_0^a dy \int_1^b \frac{1}{z^2}\, dz = a(1 - \frac{1}{b})$.

(2) Let $r = \sqrt{1 + \frac{a^2}{2}}$. First assume $b \geq r$. Then

$$\begin{aligned}
\mathrm{Area}(R_2(a,b)) &= \iint_{R_2(a,b)} \frac{1}{z^2}\, dy\, dz = 2\int_1^r \frac{1}{z^2}\, dz \int_0^{\sqrt{r^2-z^2}} dy \\
&= 2\int_1^r \frac{\sqrt{r^2-z^2}}{z^2}\, dz = 2\left[\arctan\frac{\sqrt{r^2-z^2}}{z} - \frac{\sqrt{r^2-z^2}}{z}\right]_1^r \\
&= 2(\frac{a}{2} - \arctan\frac{a}{2}) = \eta(a).
\end{aligned}$$

When $b < r$, $R_2(a,b) = R_2(a,\infty) - R'$, where R' is the subregion of $R_2(a,\infty)$ above the line $z = b$. The transformation $(y,z) \to (y/b, z/b)$ is a hyperbolic isometry, which maps R' to the region $R_2(\frac{\sqrt{r^2-b^2}}{b}, \infty)$, so the result follows from the above.

(3) From the definition it is clear that $\mathrm{Area}(R_2(a,b))$ is an increasing function of both a and b. Since $a \geq 3\pi$ and $b^2 \geq 5^2 > 1 + (1.5\pi)^2 = 1 + a^2/4$, by (2) we have

$$\mathrm{Area}(R_2(a,b)) \geq \mathrm{Area}(R_2(3\pi,b)) = 3\pi - 2\arctan\frac{3\pi}{2} > 2\pi.$$

□

The hyperbolic metric on $\mathbb{H}^3$ induces a Euclidean metric on the Euclidean plane $P = \partial\mathbb{H}^3_1$. Recall from [Th1] that a hyperbolic cusp N of toroidal type is isometric to $\mathbb{H}^3_1/G$ for some Euclidean translation group G of P of rank 2. Denote by T the boundary torus of N, and by N^b the truncated cusp $\mathbb{H}^3_{1,b}/G$. We allow $b = \infty$, in which case $N^b = N$.

If γ is a nontrivial closed curve on T, then there is a totally geodesic annulus A_γ in N^b perpendicular to the boundary, such that $A_\gamma \cap T$ is homotopic to γ. More explicitly, up to rotation and translation of $\mathbb{H}^3_1$ we may assume that γ lifts to an arc on $P = \partial\mathbb{H}^3_1$ with both endpoints on the y-axis. Let A'_γ be the annulus obtained from $R_1(t(\gamma), b)$ by identifying the two vertical lines. Then the quotient map q from $\mathbb{H}^3_{1,b}$ to N^b induces a map on A'_γ, which we define as the surface A_γ in N^b. Notice that if $\gamma = k\beta$ in $H_1(T)$, then A_γ is a k-fold cover of A_β. By Lemma 3.1(1) we have

$$\mathrm{Area}(A_\gamma) = \mathrm{Area}(R_1(t(\gamma), b) = t(\gamma)(1 - \frac{1}{b}).$$

Let F be a surface of type S in N or $\mathbb{H}^3$. We would like to estimate the area of F. Consider the 2-form

$$\omega = \frac{1}{z^2}\, dy \wedge dz$$

on $\mathbb{H}^3$. Notice that its restriction to $\mathbb{H}^2$ is the standard volume form $\omega_{\mathbb{H}^2}$ of $\mathbb{H}^2$, and if we denote by $p : \mathbb{H}^3 \to \mathbb{H}^2$ the Euclidean orthogonal projection $p(x, y, z) = (y, z)$, then $\omega = p^*(\omega_{\mathbb{H}^2})$. Therefore if $F : S \to \mathbb{H}^3$ is a surface in $\mathbb{H}^3$ then

$$\mathrm{Area}(p \circ F) = \int_S |(p \circ F)^*(\omega_{\mathbb{H}^2})| = \int_S |F^*(\omega)| = \int_F |\omega| \geq |\int_F \omega\,|$$

The map p is area non-increasing, so the area of F is at least that of $p \circ F$. In fact, more is true. Recall that ω_F denote the volume form of F induced by the Riemannian metric of $\mathbb{H}^3$.

Lemma 3.2. *Let $F : S \to \mathbb{H}^3$ be a surface in $\mathbb{H}^3$. Let $\theta(p)$ be the angle between the normal vector of F at a regular point p and the positive x-axis. Then $F^*(\omega) = \cos\theta(p)\, \omega_F$. In particular, if F is a Euclidean planar surface in $\mathbb{H}^3$ (so θ is a constant), then*

$$\int_F \omega = (\cos\theta)\mathrm{Area}(F).$$

Proof. Let (u, v) be a local coordinate system at a regular point p. Then $\mathbf{n} = F_u \times F_v$ is a normal vector of T_pF. Put $F = (x(u, v), y(u, v), z(u, v))$, and $\mathbf{n} = n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$. Then $n_1 = y_u z_v - y_v z_u$ and $\cos\theta(p) = n_1/||\mathbf{n}||$. Use $\mathbf{a} \cdot \mathbf{b}$ to denote the dot product of two vectors in $\mathbb{R}^3$. Then $g_{11} = \langle F_u, F_u\rangle = \frac{1}{z^2} F_u \cdot F_v$. Similarly for the other g_{ij}. Thus

$$\det(g_{ij}) = \frac{1}{z^4}[(F_u \cdot F_u)(F_v \cdot F_v) - (F_u \cdot F_v)^2] = \frac{1}{z^4}||F_u \times F_v||^2.$$

Hence

$$\omega_F = \sqrt{\det(g_{ij})}\, du \wedge dv = \frac{1}{z^2}||F_u \times F_v||\, du \wedge dv = \frac{||\mathbf{n}||}{z^2}\, du \wedge dv.$$

On the other hand, we have

$$\begin{aligned} F^*(\omega) &= F^*(\frac{1}{z^2}\, dy \wedge dz) = \frac{1}{z^2}(y_u du + y_v dv) \wedge (z_u du + z_v dv) \\ &= \frac{1}{z^2}(y_u z_v - y_v z_u)\, du \wedge dv = \frac{n_1}{z^2}\, du \wedge dv = \cos\theta(p)\, \omega_F. \end{aligned}$$

□

Now consider a hyperbolic cusp $N = \mathbb{H}^3_1/G$, with torus boundary $T = \partial N$. Clearly ω is invariant under Euclidean translations, hence it induces a 2-form ω_N on N. Suppose $F : S \to N$ is a surface in N. Since Lemma 3.2 is a local property, we still have

$$F^*(\omega_N) = \cos\theta(p)\ \omega_F$$

where $\theta(p)$ is the angle between the normal vector of F at p and a vector in T_pN whose lifting to $\mathbb{H}^3_1$ points to the positive x-axis direction.

Lemma 3.3. *Let $F, F_1, F_2 : S \to N^b$ be compact, oriented surfaces in N^b with boundary on ∂N^b. Then*

(1) $\mathrm{Area}(F) = \int_S \omega_F \geq \int_F |\omega_N| \geq |\int_F \omega_N|$;

(2) If $[F_1] = [F_2] \in H_2(N^b, \partial N^b)$, then $\int_{F_1} \omega = \int_{F_2} \omega$;

(3) If $[\partial F \cap T] = [\gamma] \neq 0 \in H_1(T)$ and δ is a geodesic arc on T which lifts to an arc on $\partial \mathbb{H}^3_1$ parallel to the x-axis, then $|\int_F \omega_N| = t_\delta(\gamma)(1 - \frac{1}{b})$.

(4) If $[\partial F \cap T] = [\gamma] \neq 0 \in H_1(T)$, then $\mathrm{Area}(F) \geq t(\gamma)(1 - \frac{1}{b})$.

Proof. (1) We have

$$\int_S \omega_F = \int_S |\omega_F| \geq \int_S |\cos\theta(p)\, \omega_F| = \int_S |F^*(\omega_N)| = \int_F |\omega_N|.$$

(2) Notice that

$$d(\omega) = d\,(\frac{1}{z^2}\, dy \wedge dz) = \frac{-2}{z^3}\, dz \wedge dy \wedge dz = 0,$$

so ω is a closed form. Since ω_N is induced from ω, it is also a closed form. Denote by $\overline{F}_2$ the surface F_2 with orientation reversed. The assumption means that there is a surface F_3 on ∂N^b, such that $\hat{F} = F_1 \cup \overline{F}_2 \cup F_3$ is a closed oriented surface which is null homologous in N^b. Therefore there is an oriented 3-manifold W and a map $f : W \to N^b$ with $f|_{\partial W} = \hat{F}$. By Stokes theorem, we have

$$\begin{aligned} \int_{\hat{F}} \omega_N &= \int_{\partial W} f^*(\omega_N) = \int_W d(f^*(\omega_N)) \\ &= \int_W f^*(d\,\omega_N)) = \int_W 0 = 0. \end{aligned}$$

Since F_3 lifts to a horizontal planar surface in $\mathbb{H}^3$, by Lemma 3.2

$$\int_{F_3} \omega_N = (\cos\frac{\pi}{2})\, \mathrm{Area}(F_3) = 0.$$

Therefore

$$0 = \int_F \omega_N = \int_{F_1} \omega_N + \int_{\overline{F}_2} \omega_N + \int_{F_3} \omega_N = \int_{F_1} \omega_N - \int_{F_2} \omega_N$$

and the result follows.

(3) F is homologous to the surface A_γ defined above, which lifts to a region on a vertical plane in $\mathbb{H}^3_1$. Let θ' be the angle between δ and γ. Then the acute angle θ between the normal vector of A_γ and the x-axis satisfies $\theta = |(\pi/2) - \theta'|$. Hence by the definition of $t_\delta(\gamma)$ is section 1, we have $\cos\theta = |\sin\theta'| = t_\delta(\gamma)/t(\gamma)$. It follows from (2) and Lemma 3.2 that

$$|\int_F \omega_N| = |\int_{A_\gamma} \omega_N| = (\cos\theta)\,\mathrm{Area}(A_\gamma) = (\cos\theta)\, t(\gamma)(1-\frac{1}{b}) = t_\delta(\gamma)(1-\frac{1}{b}).$$

(4) Choose a coordinate system of $\mathbb{H}^3$ so that the geodesic γ' homotopic to γ lifts to the y-axis. Let δ be an arc perpendicular to γ'. Then

$$\mathrm{Area}(F) \geq |\int_F \omega_N| = t_\delta(\gamma)(1-\frac{1}{b}) = t(\gamma)(1-\frac{1}{b}).$$

□

Lemma 3.4. *Let β be an arc on T, and let α be the geodesic segment in N homotopic to β. Let F be a compact, oriented surface in N^b such that $\partial F \cap \mathrm{Int}N^b = \alpha \cap \mathrm{Int}N^b$. Put $[\gamma] = [(\partial F \cap T) \cup \beta] \in H_1(T)$.*

(1) If $[\gamma] = 0$, then $\mathrm{Area}(F) \geq \mathrm{Area}(R_2(t(\alpha), b)))$.

(2) If $[\gamma] \neq 0$, then $\mathrm{Area}(F) \geq t_\delta(\gamma)(1-\frac{1}{b}) - t_\delta(\alpha)$ *for all slopes δ on T.*

Proof. Without loss of generality we may assume that β is a geodesic on T. Choose a coordinate system of $\mathbb{H}^3$ so that the lifting of β is an arc on the y-axis, and is symmetric about the z-axis. Let R be the image of $R_2(t(\alpha), b)$ in N^b under the projection map. Then $\hat{F} = F \cup R$ is a properly embedded surface in N^b. Notice that $(\partial\hat{F}) \cap T = (\partial F \cap T) \cup \beta$, so it is homologous to γ.

In case (1), the surface $\hat{F}$ is null-homologous in $H_2(N^b, \partial N^b) \cong H_1(T)$, hence by Lemma 3.3(2) we have $\int_{\hat{F}} \omega_N = 0$. Thus $\int_F \omega_N = -\int_R \omega_N$. Since R lifts to the region $R_2(t(\alpha), b)$ on $\mathbb{H}^2$, we have

$$\mathrm{Area}(F) \geq |\int_F \omega_N| = |\int_R \omega_N| = \mathrm{Area}(R_2(t(\alpha), b)).$$

In case (2), rechoose the coordinate system so that the geodesic on T homotopic to δ lifts to the x-axis. Then R lifts to a surface $\tilde{R}$ which is a rotation of $R_2(t(\alpha), b)$ by an angle θ. As in the proof of Lemma 3.3(3), we have $\cos\theta = t_\delta(\alpha)/t(\alpha)$, so by Lemma 3.2 we have

$$\int_R \omega_N = (\cos\theta)\,\mathrm{Area}(R) = \frac{t_\delta(\alpha)}{t(\alpha)}\mathrm{Area}(R_2(t(\alpha), b)).$$

By Lemma 3.3(3), $|\int_{\hat{F}} \omega_N| = t_\delta(\gamma)(1 - 1/b)$. Therefore,

$$\begin{aligned}\text{Area}(F) &\geq |\int_F \omega_N| \geq |\int_{\hat{F}} \omega_N - \int_R \omega_N| \geq |\int_{\hat{F}} \omega_N| - |\int_R \omega_N| \\ &\geq t_\delta(\gamma)(1 - \frac{1}{b}) - \frac{t_\delta(\alpha)}{t(\alpha)}\text{Area}(R_2(t(\alpha), b)) > t_\delta(\gamma)(1 - \frac{1}{b}) - t_\delta(\alpha).\end{aligned}$$

The last inequality is because by Lemma 3.1(2) we have $\text{Area}(R_2(t(\alpha), b)) < t(\alpha)$. □

§4. Nontrivial curves in negatively curved manifolds

Let M be a complete hyperbolic manifold of finite volume. Let N be a set of mutually disjoint cusps of M. Let $M_0 = M - \text{Int}N$. Put $T = \partial M_0 = \partial N$, which is a union of tori. We may choose N so that T lifts to a set of horospheres in $\mathbb{H}^3$, hence it has a Euclidean metric induced by the hyperbolic metric of M.

A geodesic arc α in M with endpoints on T is said to be of *type I* if a neighborhood of $\partial\alpha$ lies in M_0, and of *type II* if $\alpha \subset N$. Notice that a geodesic arc may be neither of type I nor of type II, but we will not consider such arcs.

Theorem 4.1. *Let M be a complete hyperbolic 3-manifold, and let M_0, N be as above. If $\alpha = \alpha_1 \cup ... \cup \alpha_{2p}$ is a closed curve such that (i) each α_{2i+1} is a geodesic arc of type I, (ii) each α_{2i} is a proper arc in N, and (iii) $t(\alpha_{2i}) \geq 2\pi$ for $i < p$, then α is nontrivial in M.*

Moreover, if each α_{2i-1} has both endpoints perpendicular to T, then (iii) can be replaced by (iii') $t(\alpha_{2i}) \geq \pi$ for $i < p$.

Proof. If the theorem were not true, we can choose a curve α as in the theorem, such that α is null-homotopic in M, and p is minimal among all such curves. By a homotopy we may assume that all α_{2i} are geodesics in N, so they are type II arcs. Now α is a piecewise geodesic curve with $2p$ corners, so its total external angle is less than $2p\pi$. By Proposition 2.2, it bounds a surface $F : D^2 \to M$ of disk type, such that $\text{Area}(F) < \Theta - 2\pi$, where Θ is the total external angle of α.

By a small perturbation we may assume that F is transverse to T. Then $A = F \cap T$ is a compact 1-manifold in F. Recall our convention that we will treat F the same way as an embedded surface. Thus for example A is really the restriction of F on the 1-manifold $F^{-1}(T)$ in D, and by a disk cut off by a component of A we really mean the restriction of F to a disk in D cut off by the corresponding component of $F^{-1}(T)$.

We claim that each arc component of A is outmost in the sense that it cuts off a disk Δ on F containing no other *arc* components of A. (Note that Δ could contain

some circle components of A.) Assuming otherwise, let c be a component which is not outmost. Now c is an arc on T, whose boundary cuts α into two arcs α' and α''. One of α' and α'', say α', does not contain α_{2p}, so $\alpha' \cup c$ satisfies the condition of the theorem with smaller p, and is null-homotopic in M because it bounds a subdisk of F. This contradicts the minimality of p, completing the proof of the claim.

Now let Δ be an outmost disk cut off by an arc component c of A. Then $\Delta \cap \partial D$ is one of the arcs α_i in α. We have assumed above that α_i is a geodesic, so α_i being homotopic to the arc c on T implies that α_i is in N, that is, i is an even number. Hence we can label the outmost disks as $\Delta_1, ..., \Delta_p$, with $\Delta_i \cap \partial D = \alpha_{2i}$.

Recall that Δ_i may contain some circle components of A. Let Q be the component of Δ_i cut along A which contains $\partial \Delta_i$. Since M is hyperbolic, N is π_1-injective in M, hence each boundary component of Q is null-homotopic in N because it bounds a disk in M. Let β be an arc on T homotopic to α_{2i}. Then $(\partial Q \cap T) \cup \beta = (\partial Q - \alpha_{2i}) \cup \beta$ is null-homologous on T because β is homotopic to the arc component of $\partial Q \cap T$ and the circle components of $\partial Q \cap T$ are also null-homotopic. Therefore, by Lemma 3.4(1) (with $b = \infty$) and Lemma 3.1(2), for each $i < p$ we have

$$\mathrm{Area}(\Delta_i) \geq \mathrm{Area}(Q) \geq \mathrm{Area}[R_2(t(\alpha_{2i}), \infty)] = t(\alpha_{2i}) - 2 \arctan \frac{t(\alpha_{2i})}{2}$$

for all $i < p$. Notice that $\arctan(t(\alpha_{2i})/2)$ is the angle between α_{2i} and T. Denote by θ_i the external angle at the corner between α_i and α_{i+1}. Since a_j are of type I for odd j, we have $\theta_{2i-1}, \theta_{2i} \leq \pi - \arctan(t(\alpha_{2i})/2)$, so the above inequality together with the assumption $t(\alpha_{2i}) \geq 2\pi$ implies that $\mathrm{Area}(\Delta_i) \geq \theta_{2i-1} + \theta_{2i}$ for $i < p$. Therefore

$$\mathrm{Area}(F) > \sum_{i=1}^{p-1} \mathrm{Area}(\Delta_i) \geq \sum_{j=1}^{2p-2} \theta_j > \Theta - 2\pi.$$

Since F is chosen to have area less than $\Theta - 2\pi$, this is a contradiction.

If all α_{2i-1} have endpoints perpendicular to T, then $\theta_i \leq \frac{\pi}{2} - \arctan(t(\alpha_{2i})/2)$, so the assumption $t(\alpha_{2i}) \geq \pi$ suffices to lead to a contradiction. □

We now consider Dehn fillings on M. Recall that N is a set of disjoint cusps, and $M_0 = M - \mathrm{Int} N$.

Assume $t(\gamma_i) > 2\pi + 1$ for each i. Choose b_i so that the geodesic curve γ'_i on $T'_i = \partial N_i^{b_i} - T_i$ isotopic to γ_i in N_i has length $2\pi + 1$. Choose a coordinate for $\mathbb{H}^3$ so that the geodesic on T_i homotopic to γ_i lifts to the y-axis. Then the upper edge of

$R_1(t(\gamma_i), b_i)$ is projected to γ_i'. Since the upper edge has hyperbolic length $t(\gamma_i)/b_i$, we have

$$b_i = \frac{t(\gamma_i)}{2\pi + 1}. \tag{4-1}$$

Denote by $N_i(\gamma_i)$ the manifold obtained by gluing a solid torus V_i to $N_i^{b_i}$ along T_i' so that γ_i' bounds a meridian disk in V_i. Put $N^b = \cup N_i^{b_i}$, $N(\gamma) = \cup N_i(\gamma_i)$, and $M(\gamma) = M_0 \cup N(\gamma)$. The manifold $M(\gamma)$ is the Dehn filling space of M (or more precisely, of M_0) along the multiple slope γ. By the 2π-theorem of Gromov-Thurston [GT], $M(\gamma)$ has a negatively curved metric which coincides with the original hyperbolic metric in a neighborhood of $M_0 \cup N^b$. We will assume below that $M(\gamma)$, $N(\gamma)$ and $V = \cup V_i$ are endowed with such a metric. Let C_i be the core of V_i. The identity map on $M_0 \cup N^b$ extends to a homeomorphism $M \cong M(\gamma) - \cup C_i$. We will always (topologically) identify M with $M(\gamma) - \cup C_i$ in this way; in particular, each curve α in M is also a curve in $M(\gamma)$.

Lemma 4.2. *Let $K > 2\pi+1$ be a constant, and let $\gamma = (\gamma_1, ..., \gamma_n)$ be a multiple slope on T such that $t(\gamma_i) \geq K$ for all i. Let D be a surface of disk type in $M(\gamma)$ such that $\partial D \subset T$, and D is transverse to T. If ∂D is nontrivial on T, then* $\text{Area}(D \cap N^b) \geq K - (2\pi + 1)$.

Proof. Let Q be the component of D cut along T containing ∂D. If some component of $\partial Q - \partial D$ is nontrivial in T, then by induction the subdisk D' of D bounded by this curve has $\text{Area}(D' \cap N^b) \geq K - (2\pi + 1)$, and we are done. So assume that all components of $\partial Q - \partial D$ are trivial on T. If Q were in M_0 then the above would imply that ∂D is null-homotopic in M_0, contradicting the incompressibility of T in M_0. Therefore Q is contained in $N_i(\gamma_i)$ for some component N_i of N. The above assumption means that each component of $\partial Q - \partial D$ bounds a disk on T_i, hence ∂D is null-homotopic in $N_i(\gamma_i)$. Thus $[\partial Q \cap T_i] = [\partial D] = [k\gamma_i] \in H_1(T_i)$, and $k \neq 0$ because ∂D is assumed nontrivial on T_i. Hence by Lemma 3.3(4) we have

$$\text{Area}(D \cap N^b) \geq t(k\gamma_i)(1 - \frac{1}{b_i}) = |k|(t(\gamma_i) - \frac{t(\gamma_i)}{b_i}) \geq t(\gamma_i) - (2\pi + 1).$$

The last inequality follows because $k \neq 0$, and because by (4-1) we have $t(\gamma_i) = (2\pi + 1)b_i$. $\square$

Theorem 4.3. *Let $\gamma = (\gamma_1, ..., \gamma_n)$ be a multiple slope on T such that $t(\gamma_i) \geq 12\pi$ for all i. Let $\alpha = \alpha' \cup \alpha''$ be a curve in M such that either α'' is a closed geodesic and*

$\alpha' = \emptyset$, or α'' is a type I geodesic arc and α' is an arc in N. If each component β of $\alpha'' \cap N_i$ satisfies $t_\delta(\beta) \leq t_\delta(\gamma_i) - 5\pi$ for some slope δ on T, then α is nontrivial in $M(\gamma)$.

Proof. If α is a geodesic in M_0, then it remains a geodesic in the negatively curved manifold $M(\gamma)$, hence is nontrivial. (This is well known, and also follows from Lemma 2.1(3) because $K < 0$ and $\kappa = 0$.) Therefore by choosing a component of $\alpha'' \cap N$ as α' if necessary, we may always assume that α'' is a type I geodesic. Put $\alpha'' = \alpha_1 \cup ... \cup \alpha_{2p-1}$. Then α_{2j} lie in N, and α_{2j-1} are in M_0. Assume the result is false, and let α be as in the theorem so that α is null homotopic in $M(\gamma)$, and p is minimal among all such curves.

Modify α as follows. For each α_{2i} which has nontrivial intersection with the Dehn filling solid tori V_i, homotope $\alpha_{2i} \cap V_i$ to a geodesic segment α'_{2i} in V_i, and denote the resulting arc $(\alpha_{2i} \cap N^b) \cup \alpha'_{2i}$ by β_{2i}. Since $b_i = t(\gamma_i)/(2\pi + 1) \geq 12\pi/(2\pi+1) > 5$, from Figure 3.1 we see that such modification happens only if

$$t(\alpha_{2i}) > 2\sqrt{b_i^2 - 1} > 2\sqrt{24} > 3\pi.$$

Let r be the number of arcs which have been modified. Next, deform α' to a geodesic β_{2p} in $N(\gamma)$. For simplicity, write $\beta_i = \alpha_i$ for the other arcs. The curve $\beta = \beta_1 \cup ... \cup \beta_{2p}$ is now a piecewise geodesic in $M(\gamma)$ with $2r+2$ corners, and is homotopic to α. Note that from the construction all the corners are in the hyperbolic part of $M(\gamma)$.

By Proposition 2.2, β bounds a surface F of disk type in $M(\gamma)$, such that

$$\text{(4-2)} \qquad \text{Area}(F \cap N^b) < \text{Area}(F \cap (M_0 \cup N^b)) \leq (2r+2)\pi - 2\pi = 2r\pi.$$

After a small perturbation rel ∂ we may assume that F is transverse to T. Let $A = F \cap T$. Since $\partial A = \partial F \cap T = \cup \partial \beta_i$, A has exactly p arc components. As in the proof of Theorem 4.1, the minimality of p implies that each arc a_i of A is outmost on F in the sense that it cuts off a disk Δ_i with interior containing no arc components of A. We can label a_i and Δ_i such that either $\partial \Delta_i = a_i \cup \beta_{2i-1}$ for all i, or $\partial \Delta_i = a_i \cup \beta_{2i}$ for all i.

If $\Delta_i \cap \partial F = \beta_{2i-1}$ for all i, then since the geodesic arc β_{2i-1} in M_0 cannot be homotopic in M to the arc a_i on T, there must be some circle component μ_i of A in $\text{Int}\Delta_i$ which is nontrivial on T. Applying Lemma 4.3 to the disks B_i in Δ_i bounded by μ_i, we get

$$\text{Area}(F \cap N^b) \geq \sum \text{Area}(B_i \cap N^b) \geq p(12\pi - (2\pi + 1)) > 2r\pi$$

which is a contradiction to (4-2).

Now assume $\partial\Delta_i = a_i \cup \beta_{2i}$ for all i. Consider a Δ_i such that $\beta_{2i} \neq \alpha_{2i}$, $i < p$. Recall from the definition of β_i that there are exactly r such arcs. We have shown that in this case $t(\alpha_{2i}) > 3\pi$ and $b_i > 5$, and we want to show that $\mathrm{Area}(\Delta_i \cap N^b) \geq 2\pi$. This follows from Lemma 4.2 if some circle component of A in Δ_i is nontrivial on T. Hence assume that all circle components of A in Δ are trivial on T. In particular, if we denote by Q the component of Δ_i cut along A which contains $\partial\Delta_i$, then $\partial Q - \partial\Delta_i$ is null-homotopic on T_i, so $\partial\Delta_i = \alpha_i \cup \beta_{2i}$ is also null-homotopic in $N_i(\gamma_i)$. Let β' be an arc on T_i homotopic to β_{2i} in N_i. Then $\beta' \cup a_i$ is null-homotopic in $N_i(\gamma_i)$, hence $[(\partial Q \cap T_i) \cup \beta'] = [a_i \cup \beta'] = k[\gamma_i] \in H_1(T_i)$ for some k. We can now apply Lemma 3.4 to the surface $Q = \Delta_i \cap N_i^b$: If $k = 0$ then

$$\mathrm{Area}(Q) > \mathrm{Area}(R_2(t(\beta'), b_i)) > 2\pi.$$

The last inequality follows from Lemma 3.1(3) because we have shown that $t(\beta') = t(\alpha_{2i}) > 3\pi$ and $b_i > 5$. If $k \neq 0$, choose a slope δ as in the statement of the theorem. Then by (4-1) and Lemma 4.3 we have

$$\begin{aligned}\mathrm{Area}(Q) &\geq t_\delta(k\gamma_i)(1 - \frac{1}{b}) - t_\delta(\alpha) \geq t_\delta(\gamma_i) - t_\delta(\alpha) - \frac{t_\delta(\gamma_i)}{b} \\ &\geq t_\delta(\gamma_i) - t_\delta(\alpha) - \frac{t(\gamma_i)}{b} \geq 5\pi - (2\pi + 1) > 2\pi.\end{aligned}$$

In either case, $\mathrm{Area}(\Delta_i \cap N^b) \geq \mathrm{Area}(Q) > 2\pi$. Since there are exactly r outmost disks Δ_i with $\beta_{2i} \neq \alpha_{2i}$, it follows that $\mathrm{Area}(F \cap N^b) \geq 2r\pi$, which is again a contradiction to (4-2). $\square$

5. Dehn surgery and Freedman tubing of immersed surfaces

Let M be a complete hyperbolic 3-manifold. For μ a small positive number, let $N = N_\mu$ be the toroidal cusp components of $M_{(0,\mu]}$, and $T = T_\mu = \partial N_\mu$. Let $M_0 = M - \mathrm{Int}N$. Then $M = N \cup_T M_0$.

A π_1-injective surface $F : S \to M$ is *geometrically finite* if $F_*(\pi_1 S_i)$ is a geometrically finite subgroup of $\pi_1 M \subset PSL_2(\mathbb{C})$ for each component S_i of S. We need some basic facts about geometrically finite surface groups. One is referred to [Mg] for more details.

Assume that S is connected, and $F : S \to M$ is a hyperbolic, geometrically finite surface in a complete hyperbolic manifold M. Consider the covering $p : X = X_\Gamma \to M$

corresponding to the subgroup $\Gamma = F_*(\pi_1 S)$ of $\pi_1(M)$. Then X is a geometrically finite complete hyperbolic manifold. Denote by $C(F) = C(X)$ the convex core of X, which by definition is the quotient C_Γ/Γ, where C_Γ is the convex hull of the limit set of Γ, and the action of Γ on C_Γ is induced by the action of Γ on its limit set. Since Γ contain no $\mathbb{Z}^2$ subgroup, the following is a special case of Lemma 6.5 and Theorem 6.6 of [Mg].

Lemma 5.1. *There is an $\epsilon_0 > 0$ such that if $0 < \epsilon \le \epsilon_0$, then (i) $C(X) \cap X_{[\epsilon,\infty)}$ is compact, (ii) $X_{(0,\epsilon]}$ has only finitely many components, and (iii) each component of $X_{(0,\epsilon]}$ is a $\mathbb{Z}$-cusp, which intersects $C(X)$ in a set isometric to*

$$\{(x,y,z) \in \mathbb{H}^3 \,|\, z \ge 1 \text{ and } A_1 \le y \le A_2\}/(g),$$

where g is a translation in the x-direction, and A_1, A_2 are constant depending on the cusp. $\square$

The lifting of $N = N_\mu$ to X is a set of horoballs and $\mathbb{Z}$-cusps. Denote by $\tilde{N}$ the $\mathbb{Z}$-cusp components of $p^{-1}(N)$, and let $\tilde{T} = \partial \tilde{N}$. When μ is small enough, each component of $\tilde{N}$ is a component of $X_{(0,\epsilon]}$ for some $\epsilon \le \epsilon_0$, so we can define $\mu(F)$ to be the maximum μ such that this property holds. Below we will always assume that $N = N_\mu$ and $T = T_\mu$ has been chosen such that $\mu = \mu(F)$. Note that we usually assume that F has boundary on T. When we rechoose $T = T_{\mu(F)}$, we extend F add some collars to ∂F so that ∂F still lies in T. Since $\mu(F)$ depends only on the group $\Gamma = F_*(\pi_1 S)$, this will not cause a logic problem.

Let $P = \tilde{T} \cap C(X)$. By Lemma 5.1, P is a finite set of compact annuli, one for each component of $\tilde{T}$. The *width* of a component P_i of P is defined as $w(P_i) = A_2 - A_1$, where A_i are as in Lemma 5.1. Define $w(F)$ to be the maximum of $w(P_i)$ over all component P_i of P. (If F is disconnected, take the maximum over all P corresponding to all components of F.)

The core of P_i projects to a curve α_i' on T, which is a nontrivial multiple of some slope α_i on T, usually called a parabolic slope of F. Since $\pi_1 X = \pi_1 F$, α_i' is homotopic to a nontrivial curve on F, hence a parabolic slope is a coannular slope. The reverse is also true: If a nontrivial curve α_i' on T is homotopic to a curve on F, then it represents a parabolic element of $\pi_1 F$, so its lifting on X is homotopic into some $\mathbb{Z}$-cusp, hence homotopic to some nontrivial curve on some P_i. Therefore, the set of parabolic slope of F are the same as the set of coannular slopes of F on T. By Lemma 5.1, T has only finitely many coannular slopes of F. The following theorem says that if the Dehn

filling slope is far away from all coannular slopes of F, then F remains π_1-injective after Dehn filling.

Theorem 5.2. *Let F be a hyperbolic, geometrically finite surface in M. Let $\gamma = (\gamma_1, ..., \gamma_n)$ be a multiple slope on T such that $t(\gamma_i) \geq 12\pi$ and $t_\beta(\gamma_i) \geq w(F) + 5\pi$ for all coannular slopes β of F. Then F is π_1-injective in $M(\gamma)$.*

Proof. We need to show that if α is a nontrivial curve on F, then it is also a nontrivial curve in $M(\gamma)$. Let $\tilde{\alpha}$ be its lifting to $X = X_F$. Then $\tilde{\alpha}$ is homotopic to a geodesic $\tilde{\alpha}'$ in the convex hull $C(X)$. The intersection of $\tilde{\alpha}$ with $\tilde{T}$ cuts $\tilde{\alpha}$ into arcs $\tilde{\alpha}_1, ..., \tilde{\alpha}_{2n}$, where $\tilde{\alpha}_{2i-1}$ lies in $X_{[\epsilon,\infty)}$, and $\tilde{\alpha}_{2i}$ on the cusps. By the choice of $T = T_{\mu(F)}$, the image of $C(X) \cap X_{[\epsilon,\infty)}$ is disjoint from the interior of N, hence the projection of $\tilde{\alpha}_i$ gives a decomposition $\alpha = \alpha_1 \cup ... \cup \alpha_{2n}$, with α_{2i} the components of $\alpha \cap N$. Each $\tilde{\alpha}_{2i}$ is homotopic to an arc lying on a strip of width at most $w(F)$ bounded by geodesics homotopic to the lifting of some coannular slope β of T, hence $t_\beta(\alpha_{2i}) \leq w(F)$. By assumption we have $t_\beta(\gamma_i) - t_\beta(\alpha_{2i}) \geq t_\beta(\gamma_i) - w(F) \geq 5\pi$. Therefore by Theorem 4.3 α is a nontrivial curve in $M(\gamma)$. $\square$

The most interesting case is when F is a closed essential surface in a compact hyperbolic manifold W. The following theorem says that when finitely many strips centered at coannular slopes and finitely many other slopes are excluded from the space of Dehn filling slopes, then F survives surgery. Note that F is not assumed to be geometrically finite.

Theorem 5.3. *Let T be a set of tori on the boundary of a compact, orientable, hyperbolic 3-manifold W. Let F be a compact essential surface in W with $\partial F \subset \partial M - T$, and let β be the set of coannular slopes of F on T. Then there is an integer K and a finite set of slopes Λ on T, such that F is π_1-injective in $W(\gamma)$ for all multiple slopes γ on T satisfying $\Delta(\gamma, \beta) \geq K$ and $\gamma_i \notin \Lambda$.*

Proof. We first assume that ∂W is a set of tori. Since W is hyperbolic and F is essential, no component of F is an annulus or torus, hence F is hyperbolic. Let M be the interior of W, which by definition has a complete hyperbolic structure. Since F is disjoint from T, it cannot be a virtual fiber, hence according to Bonahon and Thurston [B,Th1] it is automatically geometrically finite. More explicitly, assume that F is geometrically infinite and let X_F be the covering of M corresponding to the subgroup $\pi_1(F)$. Then Bonahon [B] showed that every end of X_F relative to the cusp neighborhoods is geometrically tame, while Thurston [Th1, Theorem 9.2.1]

showed that every end of X_F relative to cusp neighborhoods which is geometrically tame and geometrically infinite must either correspond to a virtual fiber for M or project to a geometrically tame and geometrically infinite end of M modulo cusp neighborhoods. Since we have assumed that ∂W is a set of tori, M has no geometrically infinite end modulo cusp neighborhoods, therefore F must be a virtual fiber, which is a contradiction.

Identify the manifold M_0 above with W, so $\partial W = T = T_{\mu(F)}$. Let Λ be the set of slopes λ on T such that $t(\lambda) < 12\pi$. For each β_i on some T_j, define $u(\beta_i) = \text{Area}(T_j)/t(\beta_i)$. Then $t_{\beta_i}(\gamma_j) = \Delta(\beta_i, \gamma_j)u(\beta_i)$. Choose K so that $K > (w(F) + 5\pi)/u(\beta_i)$ for all i. The result then follows from Theorem 5.2.

Now assume that W has some higher genus boundary components. If ∂W is compressible, then by an innermost circle outermost arc argument one can show that F can be homotoped to be disjoint from a maximal set of compressing disks D. Let W' be W cut along D. ($W' = W$ if $D = \emptyset$.) Then F is essential in W' except that it is possibly homotopic to a surface in a non-torus component of $\partial W'$. Let $\hat{W}'$ be the double of W' along the non-torus components of $\partial W'$. Denote by $\hat{F}, \hat{T}, \hat{\beta}, \hat{\gamma}$ the double of F, T, β, γ in $\hat{W}'$, respectively. By an innermost circle outermost arc argument one can show that $\hat{F}$ is π_1-injective in $\hat{W}$. Let $q : \hat{W}' \to W'$ be the obvious quotient map. If A is an annulus in $\hat{W}'$ with one boundary component on each of $\hat{F}$ and $\hat{T}$, then $q(A)$ is an annulus in W' with one boundary component on each of F and T. Hence $\hat{\beta}$ is the set of all coannular slopes of $\hat{F}$ in $\hat{W}'$. By the above, there is a number K and a set of slopes Λ' such that $\hat{F}$ is π_1-injective in $\hat{W}'(\hat{\gamma})$ when $\Delta(\hat{\gamma}, \hat{\beta}) \geq K$ and $\hat{\gamma}_i \notin \Lambda'$. Since F is π_1-injective in $\hat{F}$, we see that F is π_1-injective in $W'(\gamma)$. Since $W(\gamma)$ is obtained from $W'(\gamma)$ by adding some 1-handles, F is also π_1-injective in $W(\gamma)$. Let $\Lambda = q(\Lambda')$. Then the result follows. $\square$

We now consider Freedman tubings of essential surfaces. Let $\hat{S}$ be a surface containing S, such that $\hat{S} - S$ is a set of annuli. Then a surface $\hat{F} : \hat{S} \to M_0$ is called a *Freedman tubing* of F if $\hat{F}|_S = F$, and $\hat{F}(\hat{S} - S) \subset T$. We will use $A = \hat{F} - \text{Int}F$ to denote the restriction of $\hat{F}$ to $\hat{S} - \text{Int}S$, and call a component A_i of A a *tubing annulus.* Let δ_i be a component of ∂A_i. Then the *length* of a tube A_i is defined as

$$\ell(A_i) = \min \{ t_{\delta_i}(\alpha) \,|\, \alpha \text{ an essential arc on } A_i \}$$

Denote by $\ell(A) = \min \ell(A_i)$. Clearly, $\ell(A_i)$ would become very large when A_i wraps around the torus many times. For example, if $A_i \subset T_j$ is immersed and contains a

sub-annulus A_i' with both boundary components on the same geodesic curve of T_j, and A_i' wraps k times around T_j, then $\ell(A_i) \geq k\,\mathrm{Area}(T_j)/t(\delta_i)$.

Theorem 5.7 below says that a Freedman tubing of a geometrically finite surface is essential if the tubes are long enough. This generalizes a result of Freedman-Freedman [FF] and Cooper-Long [CL2] (see also [Li]), where the above result is proved for embedded surfaces. In most cases, one can apply Theorem 5.2 to show that it remains essential after most Dehn fillings. The assumption that F is geometrically finite is necessary: if F is geometrically infinite, then F is a virtual fiber, hence all Freedman tubings of F are inessential.

A boundary component δ_i of F can be pushed around T many times. We need a number to measure how far δ_i is away from a standard position. We would consider F to be in a standard position if its lifting to X lies in the convex core $C(X)$. Let $\tilde{\delta}_i$ be the component of $\partial \tilde{F}$ which projects to δ_i. Each $\tilde{\delta}_i$ is on some component $\tilde{T}_i$ of $\tilde{T}$, which contains a component P_i of P. Define a number $\rho(\delta_i)$ to be the minimum nonnegative number such that $\tilde{\delta}_i$ lies in a $\rho(\delta_i)$ neighborhood of P_i on $\tilde{T}_i$. Since $\tilde{\delta}_i$ is compact on $\tilde{T}_i$, such a number exists. If F is a (possibly disconnected) geometrically finite surface in M with some boundary components on $T = T_{\mu(F)}$, define

$$\rho(F) = \max\ \rho(\delta_i)$$

where the maximum is taken over all boundary components of F which is to be tubed.

Lemma 5.4. *Let α be an arc on $\tilde{T}$ with one endpoint p_1 on $\tilde{\delta}_i$ and the other endpoint on P_i. Then*

$$t_{\tilde{\delta}_i}(\alpha) \leq w(F) + \rho(F).$$

Proof. Homotope α to $\alpha_1 \cdot \alpha_2$, where α_1 is a shortest arc from p_1 to some point in P_i, and α_2 an arc in P_i. Since P_i is a strip bounded by geodesics of $\tilde{T}_i$ parallel to $\tilde{\delta}_i$, by definition we have $t_{\tilde{\delta}_i}(\alpha_1) \leq w(F)$, and $t_{\tilde{\delta}_i}(\alpha_2) \leq \rho(F)$. □

Two arcs α_1, α_2 in X with $\partial \alpha_i \subset \tilde{T}$ are *$\tilde{T}$-homotopic* if there are arcs β', β'' on $\tilde{T}$ such that $\alpha_1 \sim \beta' \cdot \alpha_2 \cdot \beta''$. Clearly this is an equivalence relation. An arc α in X is of type I if it projects to a type I arc in M.

Lemma 5.5. *Any proper essential arc α of $\tilde{F}$ is $\tilde{T}$-homotopic to a type I arc of X with endpoints on P.*

Proof. First deform α by a $\tilde{T}$-homotopy to an arc α_1 with $\partial \alpha_1 \subset P$. This is possible because each component of $\tilde{T}$ contains a component of P. Now homotope α_1 (rel ∂)

to a geodesic α_2 in X. Since $C(X)$ is a convex set, $\alpha_2 \subset C(X)$, so $\alpha_2 = \beta_1 \cdot \alpha_3 \cdot \beta_2$, where α_3 is a geodesic of type I with endpoints in P, and β_1, β_2 are (possibly empty) arcs in $C(X) \cap X_{(0,\epsilon]}$, which can be pushed into $\tilde{T}$, hence α_2 is $\tilde{T}$-homotopic to α_3. □

Lemma 5.6. *Let F be a geometrically finite surface in M. Let α be an essential arc of F with endpoints on boundary components δ_0, δ_1 of F which lie on $T = T_{\mu(F)}$. Then α is homotopic to $\beta_0 \cdot \alpha' \cdot \beta_1$, where α' is an arc of type I, and β_i are arcs on T with $t_{\delta_i}(\beta_i) \leq \rho(F) + w(F)$.*

Proof. Consider the lifting $\tilde{a}$ of α on $\tilde{F} \subset X$. By Lemma 5.5, $\tilde{\alpha}$ is homotopic to $\tilde{\beta}_0 \cdot \tilde{\alpha}' \cdot \tilde{\beta}_1$, where $\tilde{\alpha}'$ is of type I, and $\tilde{\beta}_i$ is an arc on some component $\tilde{T}_i$ of $\tilde{T}$ with one endpoint on each of $\tilde{\epsilon}_i$ and P_i. Projecting these curves into M, we get $\alpha \sim \beta_0 \cdot \alpha' \cdot \beta_1$. By Lemma 5.4, we have $t_{\delta_i}(\beta_i) = t_{\tilde{\delta}_i}(\tilde{\beta}_i) \leq \rho(F) + w(F)$. □

Recall that the wrapping number of an annulus A on a torus T is defined as

$$\text{wrap}(A) = \{\, |A \cdot p| \mid p \in T \,\}$$

where $A \cdot p$ denotes the algebraic intersection number between A and p, which is well defined for all points $p \notin \partial A$.

Theorem 5.7. *Let F be a geometrically finite surface in a compact hyperbolic 3-manifold W. Then there is a constant K such that if $\hat{F}$ is a Freedman tubing of F with wrap$(\hat{F}, F) \geq K$, then $\hat{F}$ is π_1-injective in W.*

Proof. Let M be the interior of W. By assumption M is a complete hyperbolic manifold. Identify M_0 above with W, possibly with some higher genus boundary components removed. Let $T = \partial M_0$. Clearly $\ell(\hat{F} - \text{Int}F)$ goes to infinity when wrap$(\hat{F}, F)$ approaches infinity. Choose K large enough such that when wrap$(\hat{F}, F) > K$, we have $\ell(\hat{F} - \text{Int}F) > 2(\rho(F) + w(F) + \pi)$.

We need to show that any nontrivial curve α on $\hat{F}$ is also nontrivial in M. If α is homotopic to a curve on F or $A = \hat{F} - \text{Int}F$ then α is nontrivial in M because F is π_1 injective. So assume $\alpha = \alpha_1 \cup ... \cup \alpha_{2n}$, where $\alpha_{2i-1} \subset F$ and $\alpha_{2i} \subset A$ are essential arcs. By Lemma 5.6, we have $\alpha_{2i-1} \sim \beta_{2i-1} \cdot \alpha'_{2i-1} \cdot \gamma_{2i-1}$, where α'_{2i-1} is a type I arc, and $t_{\delta'}(\beta_{2i-1})$ and $t_{\delta''}(\gamma_{2i-1}) \leq \rho(F) + w(F)$, where δ', δ'' are boundary components of F containing the endpoints of α_{2i-1}. Put $\alpha'_{2i} = \gamma_{2i-1} \cdot \alpha_{2i} \cdot \beta_{2i+1}$. Then $\alpha \sim \alpha'_1 \cdot \alpha'_2 \cdot \cdots \cdot \alpha'_{2n}$. Let δ_i be the boundary component of F containing an endpoint of α_{2i}. Then

$$\begin{aligned} t(\alpha'_{2i}) &\geq t_{\delta_i}(\alpha'_{2i}) \geq t_{\delta_i}(\alpha_{2i}) - t_{\delta_i}(\gamma_{2i-1}) - t_{\delta_i}(\beta_{2i+1}) \\ &\geq \ell(\hat{F} - \text{Int}F) - 2\rho(F) - 2w(F) > 2\pi \end{aligned}$$

Therefore by Theorem 4.1, α is a nontrivial curve in M. $\square$

6. Upper bounds on surgery distance and tubing length.

Theorems 5.3 is the best possible in the sense that there is no universal bounds on the number K in the theorem. Similarly, Theorem 5.7 is the best possible in the sense that there is no universal bound on how many time a surface need to tube around a torus boundary component in order to produce an essential surface. Assume that $\hat{F}$ is a Freedman tubing of an essential surface F, with tubes on a torus $T = \partial M_0$.

Theorem 6.1. *(i) For any constant K, there is an embedded, geometrically finite surface F in a hyperbolic manifold M, such that all Freedman tubing $\hat{F}$ of F with wrap$(\hat{F}) \leq K$ are inessential.*

(ii) For any constant K, there is a closed essential surface F in a hyperbolic manifold M, and a slope β on T, such that F' is compressible in $M(\gamma)$ for all γ with $\Delta(\gamma, \beta) \leq K$.

Proof. (1) Let S be a compact orientable surface of genus $g > K$ with a single boundary component c. Let $\alpha_1, ..., \alpha_g$ be a set of mutually disjoint nonseparating curves cutting S into a connected planar surface. By Theorem 1.1 of [WWZ], there exists a pseudo-Anosov map $\varphi : S \to S$ such that $\varphi(\alpha_i) = \alpha_{i+1}$ for $i < g$. (Note that $\varphi(\alpha_g) \neq \alpha_1$, otherwise φ would be reducible.) Let $W = S \times I - N(\alpha'_1)$, where α'_1 is the curve $\alpha_1 \times \frac{1}{2}$ in the interior of $S \times I$ isotopic to α_1. Let $M = W/((x,1) \sim (\varphi(x), 0))$. Since φ is pseudo-Anosov, it is easy to check that M is irreducible and atoroidal, and it cannot be a Seifert fiber space because $S = S \times 0$ is an essential hyperbolic surface in M disjoint from one boundary component of M. Therefore by Thurston's hyperbolization theorem for Haken manifolds [Th2], M is hyperbolic.

Let F be the disjoint union of two copies of S with opposite orientation. Then F is π_1 injective, and is not a virtual fiber because it is disjoint from one boundary component of M. Hence it is geometrically finite. Let $\hat{F}$ be a Freedman tubing of F with wrap$(\hat{F}) = k \leq K$. We want to show that $\hat{F}$ is inessential in M.

Let $\tilde{M}$ be the infinite cyclic covering of M dual to the surface S. Note that $\tilde{M}$ can be constructed by taking infinitely many copies of W, denoted by W_i ($i \in \mathbb{Z}$), and gluing the surface $S \times 1$ in W_i to $S \times 0$ in W_{i+1} using the map φ. Let X_k be the union of $W_1, ..., W_k$ in $\tilde{M}$. Then $\hat{F}$ lifts to a surface in $\tilde{M}$ homotopic to ∂X_k. Put $\alpha'_i = \alpha_i \times \frac{1}{2}$. One can check that when $k \leq K$, X_k is homeomorphic to the manifold $(S \times I) - \alpha'_1 \cup ... \cup \alpha'_k$. Let β be an essential arc on S disjoint from all α_i. Then $\beta \times I$ is a compressing disk of ∂X_k. It follows that $\hat{F}$ is compressible in M.

(2) Let $\hat{F}$ be a Freedman tubing of F such that $\hat{F}$ is essential in M, and the wrapping number w of $\hat{F}$ is minimal among all such surface. Since F is geometrically finite and embedded, the existence of such a surface follows from [CL2] or [Li], or from Theorem 5.7. Let β be the boundary slope of F, and let A be the tubing annulus $\hat{F} - \text{Int}F$. Assume that $\Delta = \Delta(\gamma, \beta) \leq K$. Notice that the annulus A is rel ∂ homotopic in the Dehn filling solid torus to another annulus A' on ∂M with wrapping number $w' = |w - \Delta|$, so $\hat{F}$ is homotopic in $M(\gamma)$ to a surface $\hat{F}' = F \cup A'$ which is a Freedman tubing of F with wrapping number w'. By the choice of w, it follows that $\hat{F}$ is inessential in $M(\gamma)$ for all γ such that $\Delta(\gamma, \beta) < 2w$. By (1) we have $2w > w > K$, hence the result follows. □

Although there is no universal upper bound on the wrapping number of an essential Freedman tubing surface, it has been shown by Li [Li] that an upper bound in terms of genus and number of boundary components of F does exist if F is an embedded surface. Li showed that if F is embedded with genus g and b boundary components, then a Freedman tubing of F is essential if its wrapping number is at least $6g+2b-3$.

Problem 6.2. *Find the minimal constant $C(g,b)$ such that if F is a geometrically finite embedded surface with genus g and b boundary components, then all Freedman tubing of F with wrapping number at least $C(g,b)$ is essential.*

Li's result [Li] shows that $C(g,b) \leq 6g+2b-3$, and the proof of Theorem 6.1 shows that $C(g,b) > g$.

For immersed surface, no such number would exist if we do not assume that F is in standard position. The reason is because we can slide one component of ∂F around the torus many times, so when tubing on the opposite direction, a long part of the tube would just homotope that boundary component of F back to its original position. However, one can consider the number of tubings which is inessential. For the embedded case, it is at most $2C(g,b)+1$. For simplicity let us consider the case that F has only two boundary components.

Conjecture 6.3. *Let F be a surface with two boundary components, both on a torus component of ∂M. Let g be the genus and b the number of boundary components of F. Then exists a constant $C'(g,b)$ depending only on g and b, such that all but at most $C'(g,b)$ of the Freedman tubings of F are π_1-injective.*

The following result gives an estimation of tubing length when F is a totally geodesic surface, which leads to an upper bound on wrapping numbers of inessential Freedman

tubing in this special case. Existence of immersed totally geodesic surfaces can be found in [AR] and [Re].

Theorem 6.4. *Let T be the boundary tori of a set of disjoint cusps N in M, let F' be a totally geodesic surface in M, and let $F = F' \cap M_0 = M - \mathrm{Int}N$. If $\hat{F}$ is a Freedman tubing of F with $\ell(\hat{F} - \mathrm{Int}F) \geq \pi$, then $\hat{F}$ is π_1-injective in M.*

Proof. Notice that we do not require $T = T_{\mu(F)}$. The intersection of F' with N is a set of totally geodesic annuli, hence they are perpendicular to $T = \partial N$. A nontrivial curve α on $\hat{F}$ can be homotoped on $\hat{F}$ either to a curve on F or to a curve $\alpha_1 \cup ... \cup \alpha_{2n}$ with α_{2i-1} a geodesic arc on F perpendicular to T, and α_{2i} an essential arc on $\hat{F} - F$. Since F is totally geodesic, α_{2i-1} is also a geodesic of M, hence the result follows from Theorem 4.1. □

Corollary 6.5. *Let F be as in Theorem 6.4. If $\hat{F}$ is an inessential Freedman tubing of F, then wrap$(\hat{F}) \leq 2\pi^2(2g+b-2)/\sqrt{3}$.*

Proof. Let A be a tubing annulus of $\hat{F}$, and let β be the boundary slope of A. Extend F to a complete hyperbolic surface F' by adding a cusp at each of its boundary component. By the Gauss-Bonnet theorem, Area$(F') = 2\pi(2g+b-2)$. Each cusp with boundary on ∂A has area $= t(\beta)$, and there are two of them, hence $t(\beta) < \pi(2g+b-2)$. Choose T to be a set of maximal cusps, then each slope of T has length at least 1, hence the area of each component of T is at least $\sqrt{3}/2$. If γ and β are on the torus T_i, then

$$\ell(A) \geq \mathrm{wrap}(A)\mathrm{Area}(T_i)/t(\beta) \geq \mathrm{wrap}(\hat{F})\sqrt{3}/2\pi(2g+b-2).$$

By Theorem 6.4, $\hat{F}$ is essential if $\ell(A) \geq \pi$ for all tubes A of $\hat{F}$, which is true if wrap$(\hat{F}) \geq 2\pi^2(2g+b-2)/\sqrt{3}$. □

If F is a closed, embedded, incompressible surface in M_0 which is not coannular to a torus $T \subset \partial M_0$, then Theorem 1 of [Wu] says that F remains incompressible in $M(\gamma)$ for all but at most three slopes γ on T. For immersed essential surfaces F in M without coannular slopes (also called accidental parabolics), Theorem 1.1 says that F remains essential in $M(\gamma)$ except for finitely many γ on T. The answer to the following problem is likely to be negative. If M is not assumed hyperbolic, there are examples showing that no upper bound exists. However, no examples are known for hyperbolic manifolds.

Problem 6.6. *Let F be a closed essential surface in a hyperbolic manifold M, and assume that F has no coannular slopes. Does there exist a universal upper bound on the number of slopes γ on a torus boundary component T of M_0 such that F is inessential in $M(\gamma)$?*

Many hyperbolic manifolds do not contain closed *embedded* essential surfaces. However, it was proved in [CLR] that any hyperbolic M with some toroidal cusps contains a closed essential surface. The surfaces constructed there are Freedman tubings of some surfaces in certain covering spaces of M, and hence all have coannular slopes. The following seems to be an interesting open problem. The corresponding problem for closed hyperbolic manifold is also open, and is part of the virtual Haken conjecture.

Conjecture 6.7. *Every hyperbolic manifold with toroidal cusps contains a closed essential surface without coannular slopes.*

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Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: wu@math.uiowa.edu